Type-Based Distributed Estimation over Multiaccess Channels

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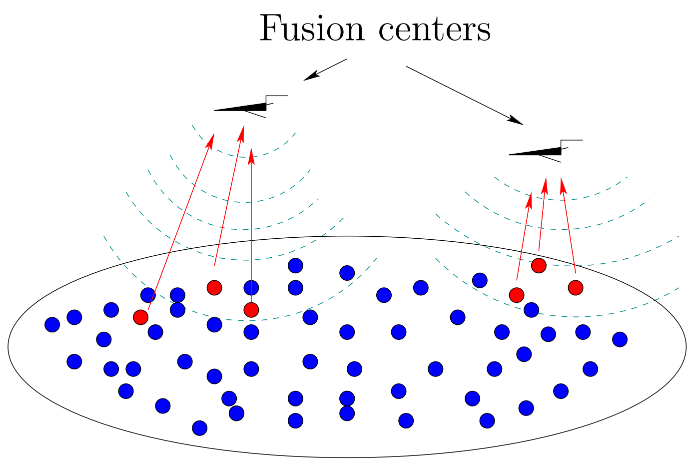
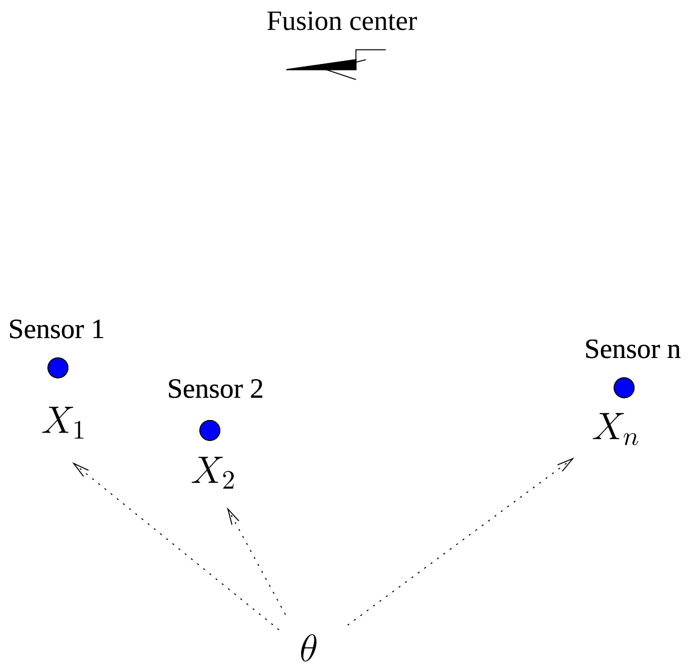
Motivation: Sensor Networks



- Sense a physical phenomena, transmit it to a fusion center.
- Fusion centers estimate the field parameters, and deliver the estimate.

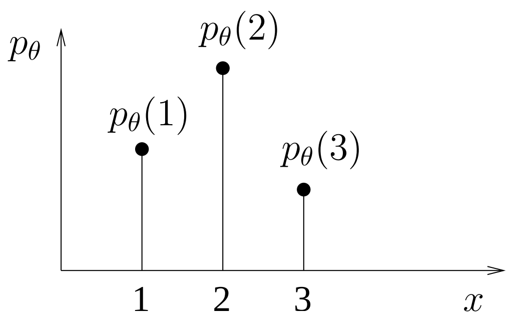
Problem Formulation



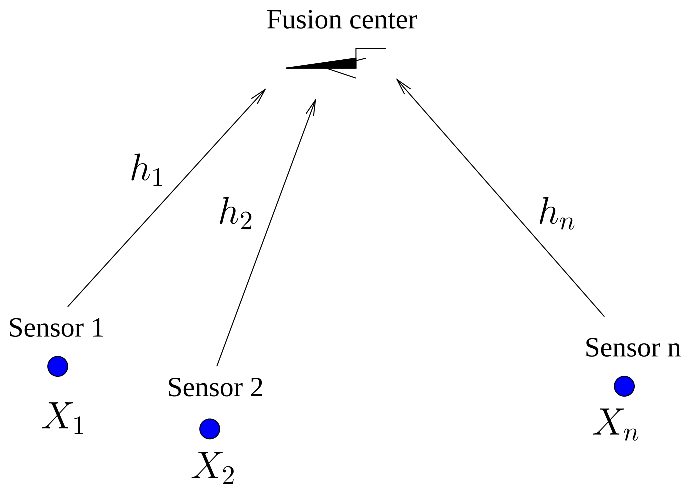
- Estimate $\theta \in \mathbb{R}$.
- Observation $X_1, \cdots, X_n$ are i.i.d. conditioned on θ.

Assumptions:

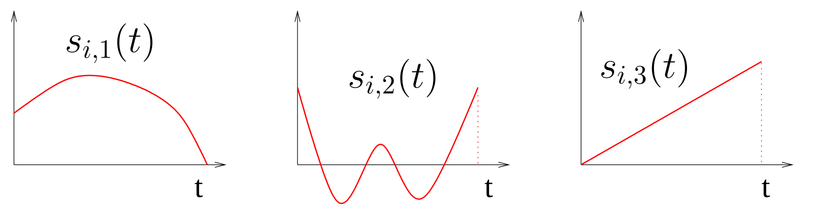
- X_i takes values in $\{1, \cdots, k\}$.
- $X_i \sim p_\theta$, where p_θ is a probability mass function.



Problem Formulation Cont'd



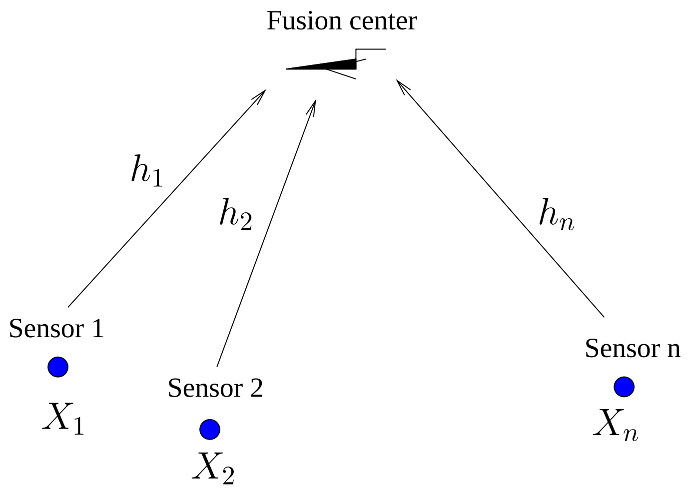
- Sensor i has a set of k channel waveforms $s_{i,1}, \cdots, s_{i,k}$.
- Upon observing X_i, it transmits s_{i,X_i}.



Received signal: $z = \sum_{i=1}^{n} h_i s_{i,X_i} + w$.

- Channel gains $h_1, \cdots, h_n \in \mathbb{R}$ are i.i.d.
- The noise is white $\mathcal{N}(0, \sigma^2)$.
- Energy constraint $||s_{i,j}||^2 \leq E$.

Problem Formulation Cont'd



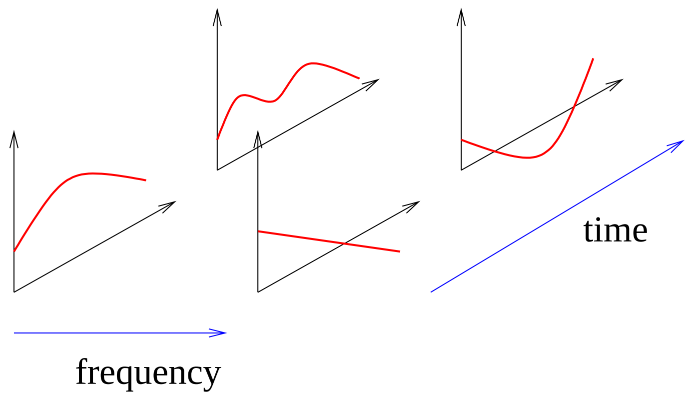
Estimator:

- The parameter θ is estimated based on the received signal z.
- $\hat{\theta}(z)$ is the estimate, $\hat{\theta}$ is the estimator.

Objective:

Design the channel waveforms and the estimator to minimize the Mean Square Error (MSE) $\mathbb{E}\{(\hat{\theta} - \theta)^2\}$.

Classical Approach



- Collision among users is “bad!”
- Solution: orthogonalize transmissions.
- Can be done by time/frequency/code division.

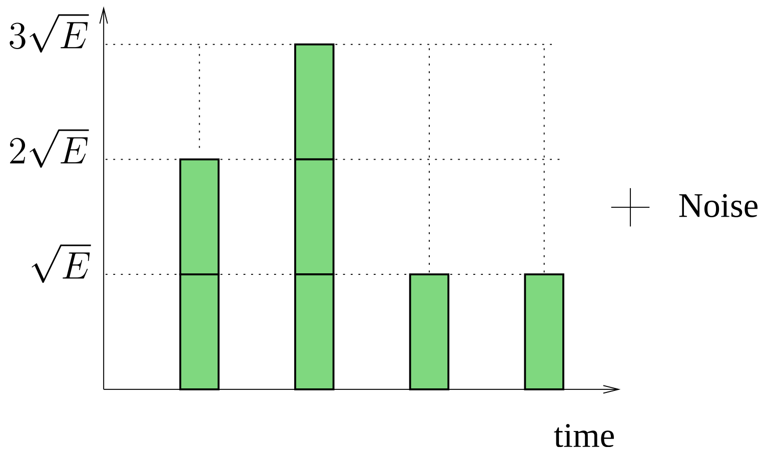
Advantages:

- Allows us to use the standard layered approach.
- Well understood. Rather easy to implement.

Caveat:

- The bandwidth requirement is significant for large n.
- Neglects the dependency among sensor data.

Proposed Approach: TBMA

z= 

- Nodes transmit simultaneously with Pulse Position Modulation (TBMA).
- $s_{i,X_i} = \sqrt{E}\delta_{X_i}$, where $\delta_1, \cdots, \delta_k$ are orthonormal pulses.
- When all $h_i = 1$,

$$z = \text{histogram+noise.}$$

Advantages:

- Uses much less bandwidth/time than orthogonal approaches.
- The MSE with TBMA is *asymptotically optimal* as $n \to \infty$.

Remark:

- Any set of orthonormal $\delta_1, \cdots, \delta_k$ can be used.

Outline

- Introduction
- Performance analysis
  - ▶ Fundamental limits
  - – TBMA with deterministic h_i
  - – TBMA with random h_i
  - – Orthogonal allocation
- Transmitter channel side information
- Conclusion

Fundamental Limits

Cramer-Rao Bound:

Let $\hat{\theta}$ be an *unbiased* estimator based on $X_1, \cdots, X_n$. Then,

$$\mathbb{E}\{(\hat{\theta} - \theta)^2\} \geq \frac{1}{nI(\theta)}, \tag{1}$$

where $I(\theta) = \sum_{i=1}^{k} \frac{(dp_\theta(i)/d\theta)^2}{p_\theta(i)}$ is the *Fisher information* in X_i.

Asymptotic Efficiency:

There exists a class of estimators based on $X_1, \cdots, X_n$ satisfying

$$\hat{\theta} \doteq \mathcal{N}(\theta, \frac{1}{nI(\theta)}), \qquad \text{for large } n. \tag{2}$$

Notation "$\doteq$" means $\hat{\theta} \xrightarrow{\text{p}} \theta$ and $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{\text{d}} \mathcal{N}(0, \frac{1}{I(\theta)})$ as $n \to \infty$.

Fundamental Limits Cont'd

Key observations:

- To achieve asymptotic efficiency, an estimator need *not* have access to all data $X_1, \cdots, X_n$.
- Knowledge of a *sufficient statistic* is actually enough.

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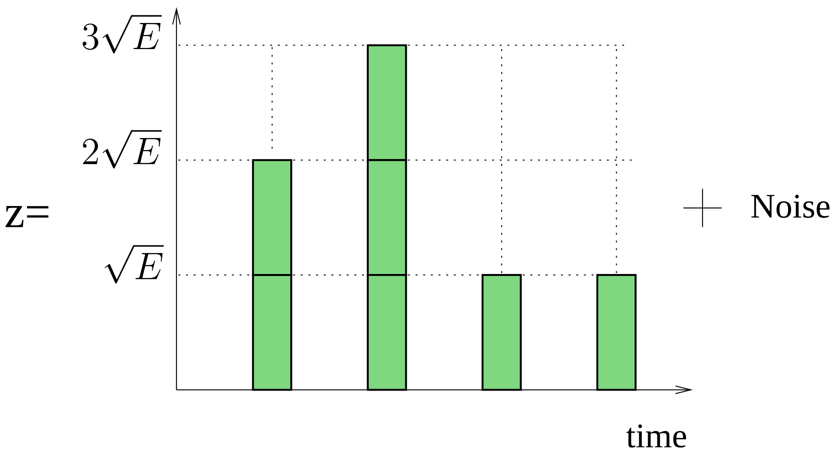
TBMA with Deterministic h_i



- A sufficient statistic is empirical measure:

$$\tilde{p} := \frac{\text{histogram}}{n}.$$

- Scale the received signal:

$$y := \frac{z}{\sqrt{E}n} = \tilde{p} + \underbrace{\mathcal{N}(0, \frac{\sigma^2}{n^2 E})}_{(*)}.$$

Questions:

- How bad is $(*)$?
- What estimator should be used?

TBMA with Deterministic h_i Cont'd

Answers:

(i) $\tilde{p} \doteq \mathcal{N}(p_\theta, \frac{1}{n}\Sigma)$ for large n, where $\Sigma = \text{Diag}(p_\theta) - p_\theta p_\theta^T$.

$$y = \tilde{p} + \mathcal{N}(0, \frac{\sigma^2}{n^2 E}) \quad \Rightarrow \quad y \doteq \mathcal{N}(p_\theta, \frac{1}{n}\Sigma).$$

(ii) Maximum-likelihood estimator (MLE) based on y is prohibitive.
Let $y = \mathcal{N}(p_\theta, \frac{1}{n}\Sigma)$, then its pdf is

$$f(y_1, \cdots, y_k \mid \theta) = \exp\left(-\frac{n\sum_{i=1}^k \frac{(p_\theta(i)-y_i)^2}{p_\theta(i)} + \log\prod_{i=1}^k p_\theta(i)}{2}\right) g(y).$$

Given y, minimize $\displaystyle\sum_{i=1}^k \frac{(p_\theta(i) - y_i)^2}{p_\theta(i)}$ for asymptotic MLE.

TBMA with Deterministic h_i Cont'd

Theorem 1: The proposed estimator $\hat{\theta}$ minimizing

$$M(\theta) := \sum_{i=1}^{k} \frac{(p_\theta(i) - y_i)^2}{p_\theta(i)} \tag{3}$$

with respect to $\theta \in \mathbb{R}$ satisfies $\hat{\theta} \doteq \mathcal{N}(\theta, \frac{1}{nI(\theta)})$ for large n.

Remarks:

- The asymptotic performance of TBMA is as if the fusion center has direct access to X_i's.
- No unbiased $\hat{\theta}$, even the ones with direct access to X_i's, can do better than this.
- The theorem holds independent of the noise power σ^2.
- The σ^2 determines the speed of convergence to the asymptotic MSE.

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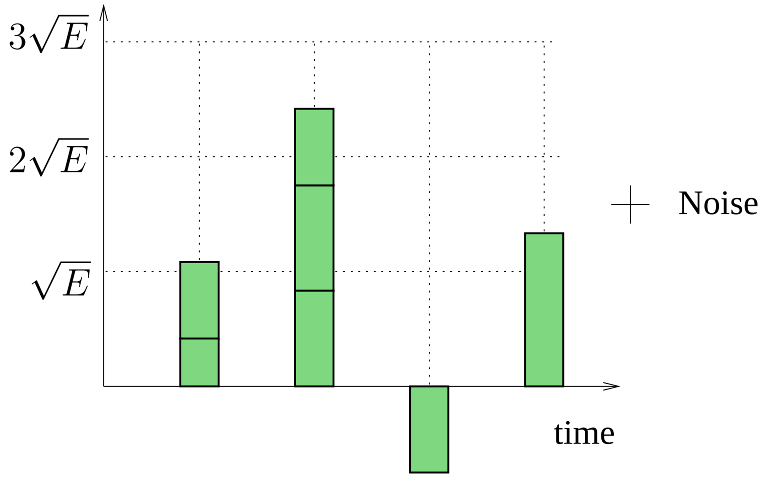
TBMA with Random h_i

z= 

- Assume h_i has *non-zero* mean $h := \mathbb{E}(h_i)$, and $\sigma_h^2 := \mathrm{Var}(h_i)$.
- Define $y = \frac{z}{\sqrt{E}hn}$.
- Observe $y \doteq \mathcal{N}(p_\theta, \frac{1}{n}\Sigma)$, where

  $\Sigma = (1 + \frac{\sigma_h^2}{h^2})\mathrm{Diag}(p_\theta) - p_\theta p_\theta^T$.

$$f(y|\theta) \propto \exp\left(-\frac{n(y-p_\theta)^T\Sigma^{-1}(y-p_\theta) + \log|\Sigma|}{2}\right).$$

TBMA with Random h_i

Theorem 2: The estimator $\hat{\theta}$ minimizing

$$M(\theta) = (y - p_\theta)^T \Sigma^{-1} (y - p_\theta), \tag{4}$$

with respect to $\theta \in \mathbb{R}$ satisfies

$$\hat{\theta} \doteq \mathcal{N}(\theta, \frac{1 + \frac{\sigma_h^2}{h^2}}{nI(\theta)}), \qquad \text{for large } n.$$

Remarks:

- The performance loss due to channel randomness is $\propto (1 + \frac{\sigma_h^2}{h^2})$.
- When $h \approx 0$, the loss is significant.
- At the extreme case $h = 0$, the MSE does not go to zero even though $n \rightarrow \infty$. This is true for all unbiased $\hat{\theta}$.

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Performance of Orthogonal Allocation

- Let $s_1, \cdots, s_k \in \mathbb{C}^m,\ ||s_i||^2 \leq 1$, be constellation points.
- Sensor i transmits the waveform corresponding to $\sqrt{E} s_{X_i}$.
- Signal received from the i'th sensor:

$$z^{(i)} = h_i \sqrt{E} s_{X_i} + v^{(i)}, \tag{5}$$

  where $v^{(i)} \sim \mathcal{N}(0, \sigma^2 I)$.
- When $h_i = 1$, the $z^{(1)}, \cdots, z^{(n)}$ are i.i.d. with *Gaussian mixture* density:

$$z^{(i)} \sim \sum_{j=1}^{k} p_j \mathcal{N}(\sqrt{E} s_j, \sigma^2 I).$$

Performance of Orthogonal Allocation Cont'd

Asymptotic Performance:

- For any unbiased $\hat{\theta}$,

$$\mathbb{E}\{(\hat{\theta}-\theta)^2\} \geq \frac{1}{nJ(\theta)}, \tag{6}$$

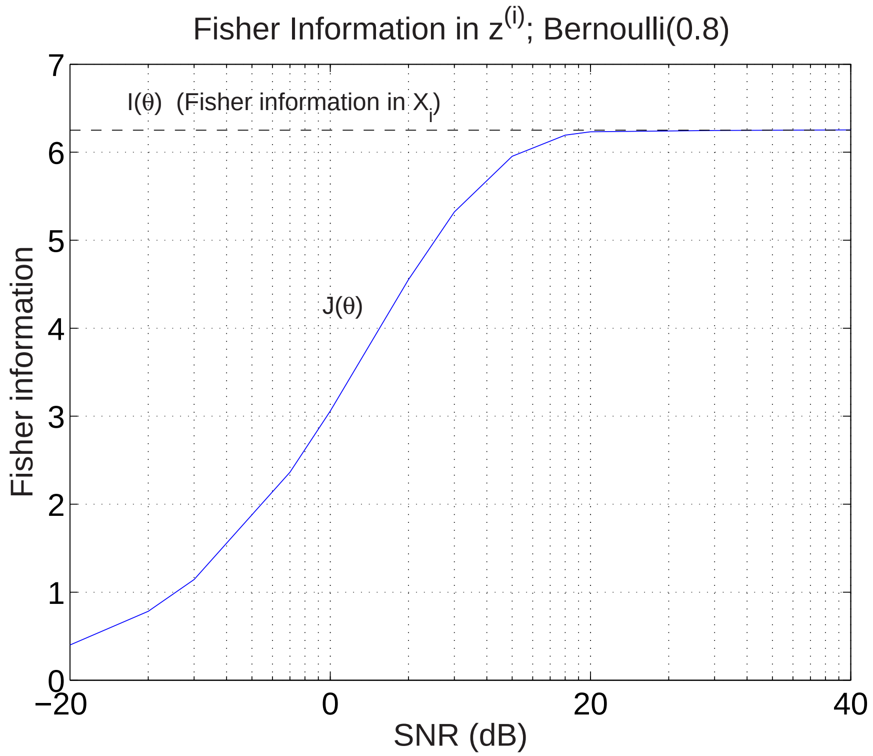
where $J(\theta) = \mathbb{E}_{z^{(i)}}\left[\left(\frac{d\log f(z^{(i)})}{d\theta}\right)^2\right]$ is the *Fisher information* in $z^{(i)}$.

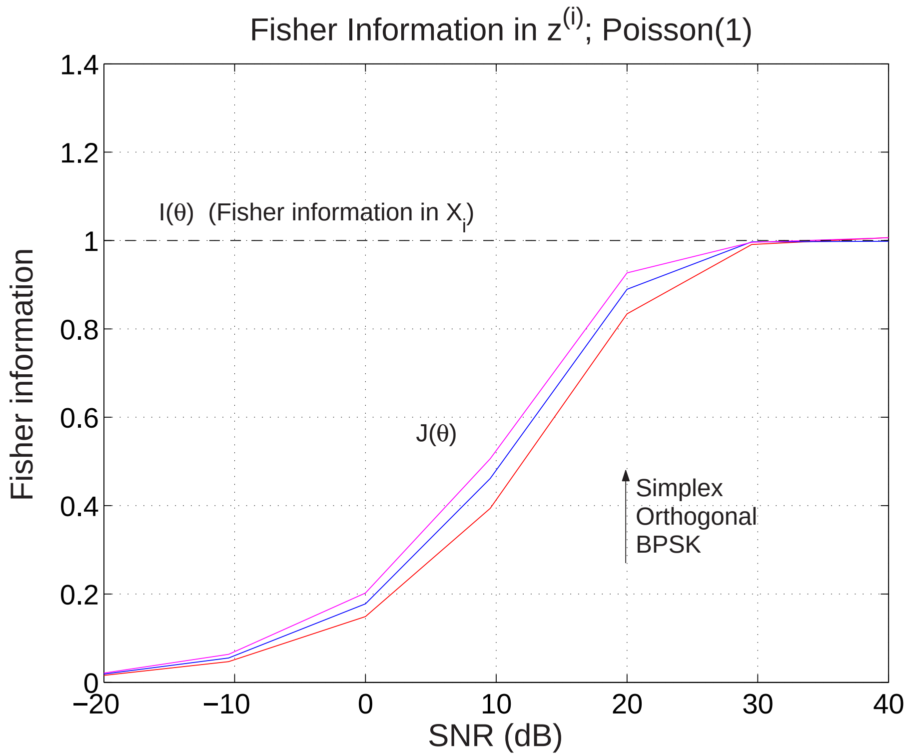
- The MLE based on $z^{(1)}, \cdots, z^{(n)}$ satisfies $\hat{\theta} \doteq \mathcal{N}(\theta, \frac{1}{nJ(\theta)})$.

Remarks:

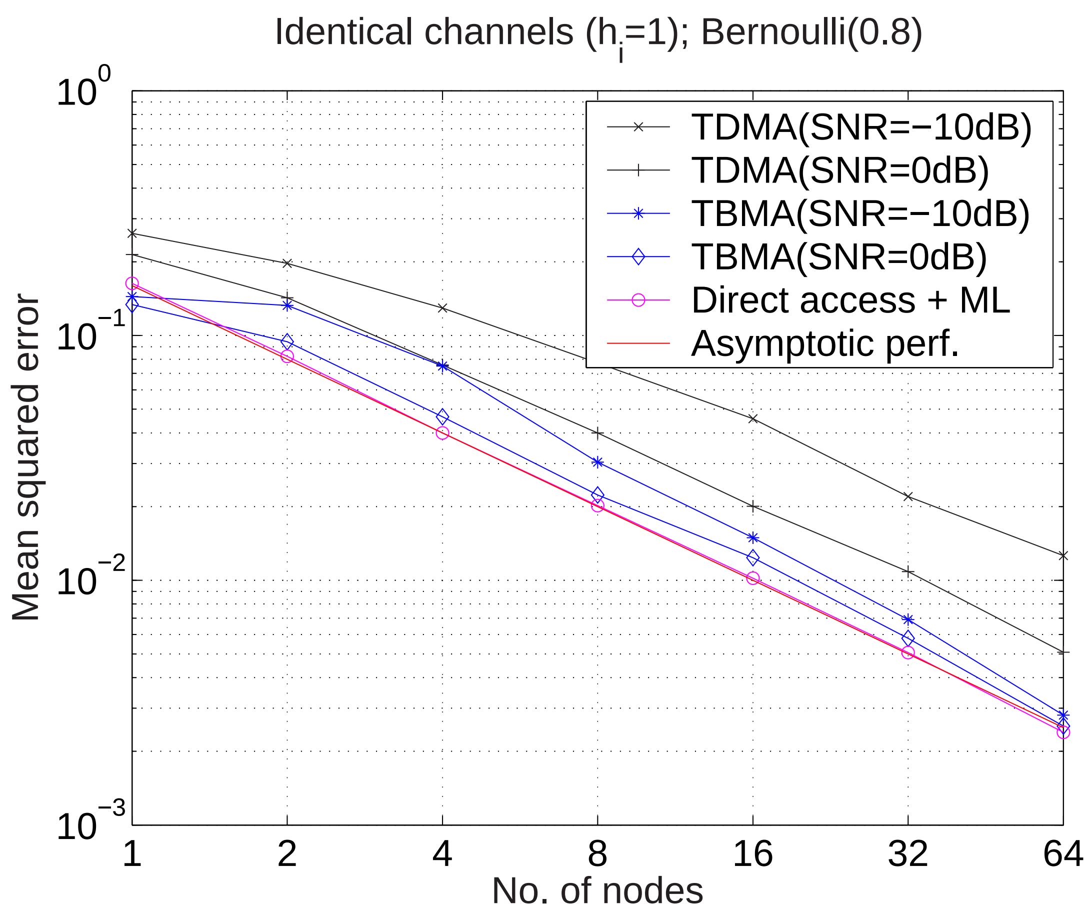
- For the best asymptotic performance $J(\theta)$ should be maximized with respect to $s_1, \cdots, s_k$.
- For $h_i = 1$, the antipodal constellation maximizes $J(\theta)$ for $k = 2$.
- In general, the optimal constellation depends on the family $\{p_\theta : \theta \in \mathbb{R}\}$.

Performance of Orthogonal Allocation Cont'd

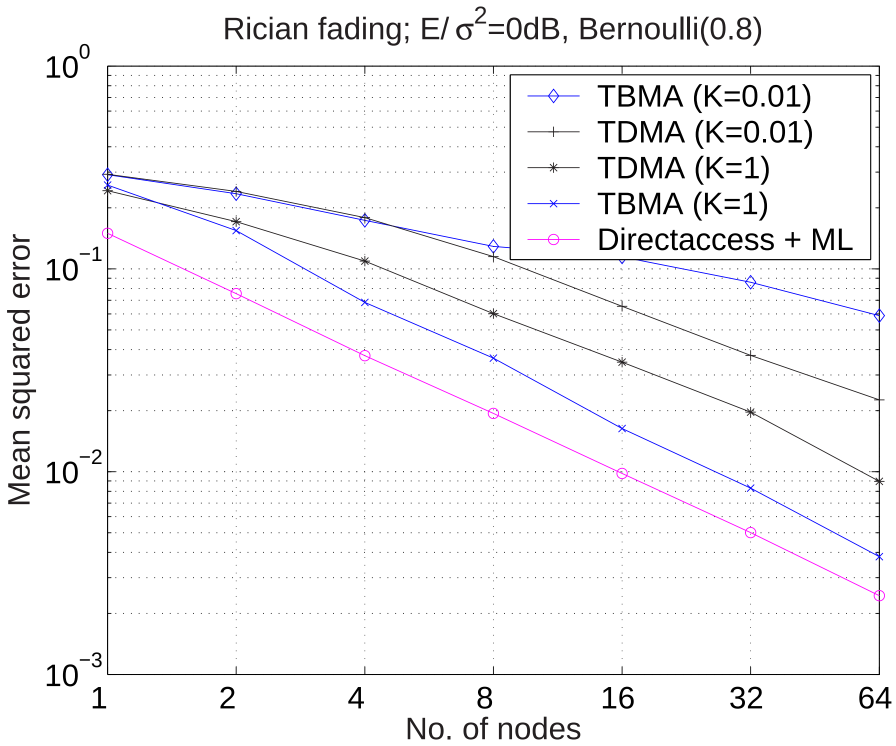




A Numerical Example



Numerical Example - 2



- The channel is Rician distributed, *i.e.*,

$$h_i = \sqrt{\frac{K}{K+1}} + \sqrt{\frac{1}{K+1}}\mathcal{CN}(0,1),$$

where $K > 0$ is a deterministic number ($K = 0 \Rightarrow$ Rayleigh).

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Channel Side Information (CSI) at the Transmitter

- In certain cases, the transmitter nodes may be able to learn their channel states before the transmission.
- Transmitter CSI can be utilized to solve the problem of zero-mean h_i.

- Let $h_i := r_i e^{j\rho_i}$.
- The i'th node transmits $P(r_i)e^{-j\rho_i}\sqrt{E}\delta_{X_i}$ in TBMA, where $P(\cdot)$ is a power control rule satisfying

$$\mathbb{E}_{r_i}[P^2(r_i)] \leq 1.$$

CSI at the Transmitter Cont'd

- One possibility is to invert the channel: $P(r) = \frac{1}{r}$. This effectively cancels out the effect of fading.
- Complete inversion may require infinite energy $\mathbb{E}_r(1/r^2)$ as in the case of Rayleigh channels.
- To circumvent this, consider the following generalization:

$$P(r) = \begin{cases} \alpha/r, & r \in [\beta, \infty) \\ \gamma, & \text{oth.} \end{cases}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants independent of r.

CSI at the Transmitter Cont'd

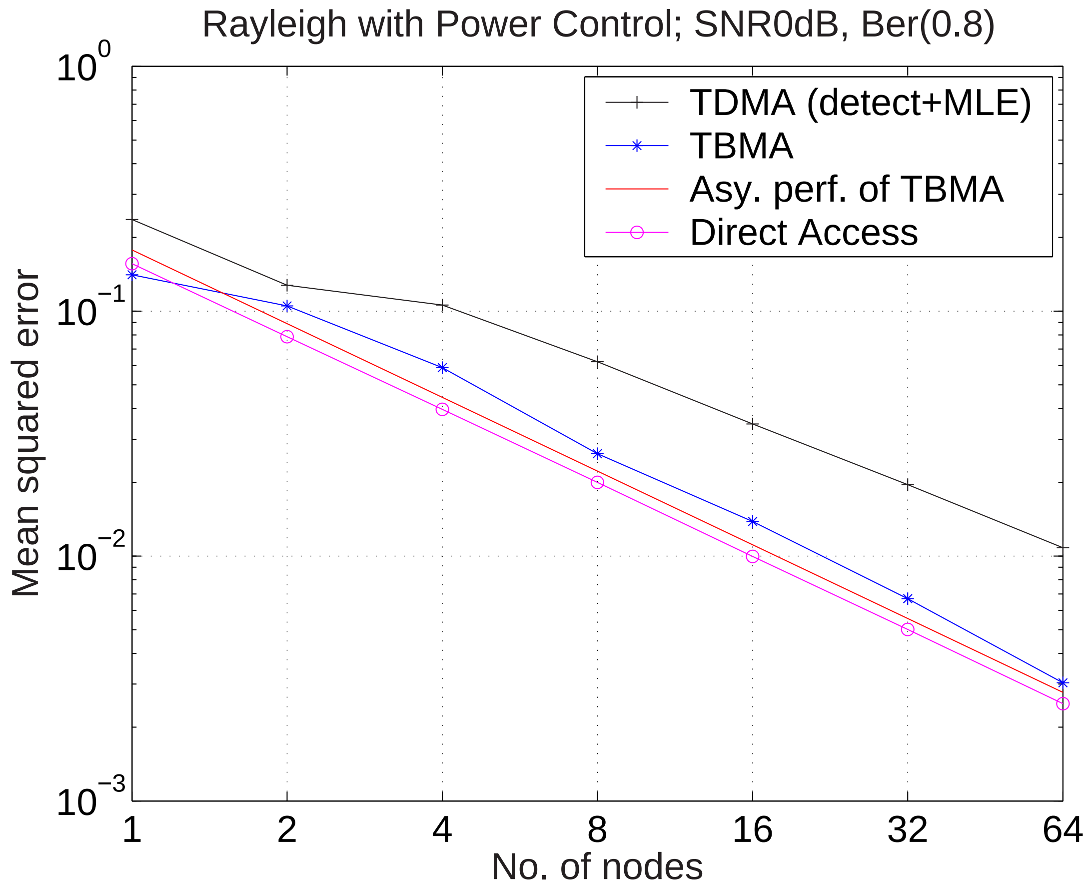
- The channel gain seen by the receiver is $\tilde{h}_i := r_i P(r_i)$.

- The asymptotic MSE is given by $(1 + \frac{\sigma^2_{\tilde{h}}}{\tilde{h}^2})/nI(\theta_0)$, where $\tilde{h} = \mathbb{E}(\tilde{h}_i)$, $\sigma^2_{\tilde{h}} = \text{Var}(\tilde{h}_i)$.

  Lemma: As α and β converge to zero,

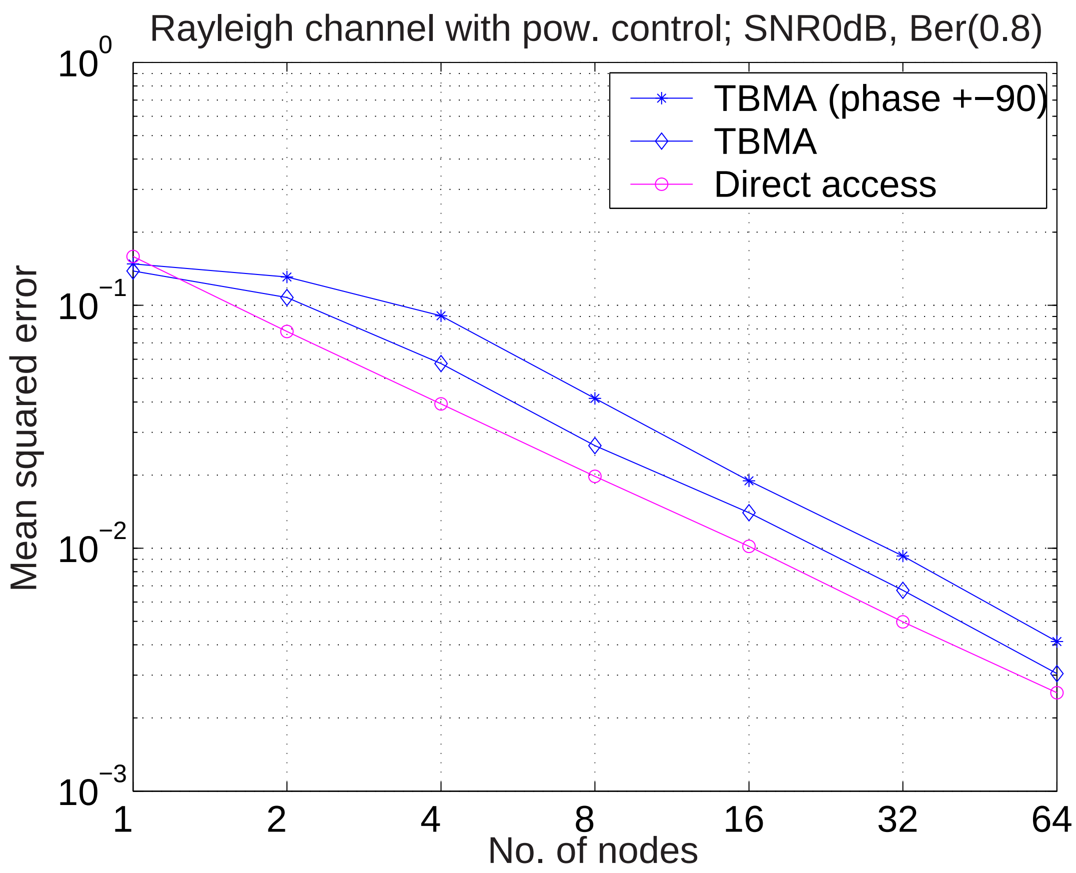
$$1 + \frac{\sigma^2_{\tilde{h}}}{\tilde{h}^2} \longrightarrow 1.$$

Numerical Example - 3

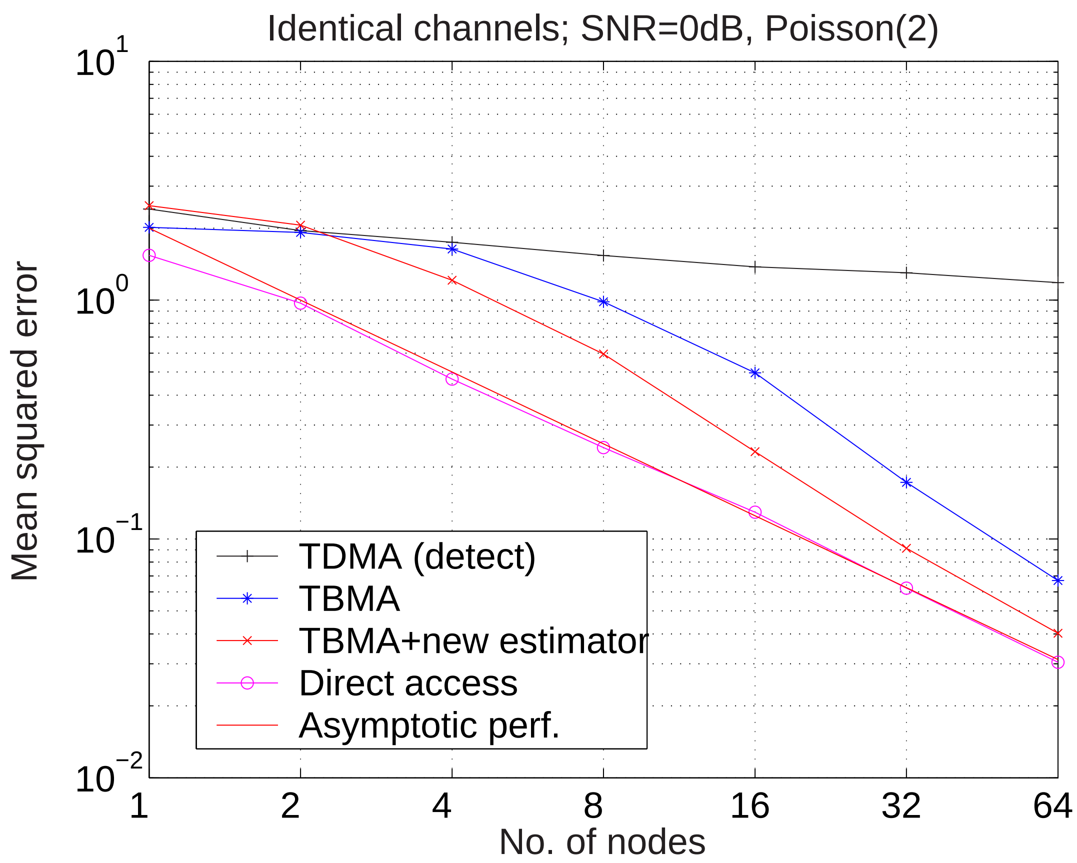


- Power controlled Rayleigh channel ($\alpha = 1, \beta = 0.89, \gamma = 1/\beta$).

Numerical Example - 4



Numerical Example - 5



Conclusions

- We considered the problem of communicating sensor readings over a Gaussian multiaccess channel.
- The TBMA scheme is proposed as an alternative to the conventional orthogonal allocation.
- We proposed an estimator and characterized its asymptotic MSE.
- The TBMA is bandwidth efficient and also has favorable MSE.
- The TBMA needs to be used with transmitter CSI in case the channel has zero mean.

Related Publications

- G. Mergen and L. Tong "Type Based Estimation Over Multiaccess Channels," submitted to *IEEE Transactions on Signal Processing*, July 2004.
- G. Mergen and L. Tong "Estimation Over Deterministic Multiaccess Channels ," *42nd Annual Allerton Conference on Communications, Control and Computing*, 2004.
- K. Liu and A. Sayeed, "Optimal Distributed Detection Strategies for Wireless Sensor Networks," *42nd Annual Allerton Conference on Communications, Control and Computing*, Monticello, IL, 2004.